10 years - ANNIVERSARY EDITION *TRAKIA JOURNAL OF SCIENCES*
Trakia Journal of Sciences, Vol. 10, No 4, pp 1-4, 2012

Available online at:
http://www.uni-sz.bg
ISSN 1313-7069 (print)
ISSN 1313-3551 (online)

Original Contribution

HEURISTIC CONTENT OF THE PLAUSIBLE REASONING AND PREDICTION IN MATHEMATICS PROBLEM SOLVING

Zh. Zhelev*

Faculty of Economics, Department of Informatics and Mathematics, Trakia University, Stara Zagora, Bulgaria

ABSTRACT
One of the most important characteristics of thinking as a process is the ability of the human mind to create an imaginable situation and to control all actions according to that situation. From here comes the possibility a human being to *predict* or to *formulate* plausible assertions.
Therefore, predictions and ability to draw out plausible hypothesis are key components in the act of mathematics problem solving. Prediction in problem solving is continuous throughout the entire act of mathematics problem solving and, on the other hand, it is the firm fundament of heuristics in general

Key words: problem solving, heuristics, plausible reasoning, prediction

2000 Mathematics Subject Classification: 97D50

1. Plausible reasoning in math problem solving – one particular example

The role of prediction in the act of problem solving is quite essential. Prediction is crucial for determination of plausible propositions and is leading when we solve algorithmic or heuristic problems [1]. Finding out the solving method is the key characteristic of the heuristic approach. To be more specific, let start with the following

Problem 1. *Prove that the sum of squares of the parts of any two intersecting perpendicular chords in a given circle is a constant.*

Search for solution: Let have a circle *k* with a center *O* and radius *r* and two chords *AB* and *CD* intersecting at the point *F* **(Fig. 1)**. The problem would become easier if we could "predict" somehow this constant. That is not difficult in this case since if $F \equiv O$ the sum of the squares is equal to $4r^2$. Now, the assertion "the sum $AF^2 + BF^2 + CF^2 + DF^2 = 4r^2$" is quite plausible because it is confirmed in the cases when *F* is close to the circle *k* as well.

The heuristic approach which is most suitable for solving this problem is to use Cartesian coordinates and parameterizations. We will try to express $AF^2 + BF^2 + CF^2 + DF^2$ by r^2 and we will make such extra constructions so some right-angle triangles to appear with sides equal to *r*.

Solution: Let *E* and *G* be the intersecting points of the perpendiculars from the center *O* to the chords *AB* and *CD* respectively. Let also $FE = OG = x$ and $FG = OE = y$. Then

$$AF^2 + BF^2 + CF^2 + DF^2 = (AE - x)^2 + (EB + x)^2 + (CG - y)^2 + (GD + y)^2 =$$
$$AE^2 - 2xAE + x^2 + EB^2 + 2xEB + x^2 + CG^2 - 2yCG + y^2 + GD^2 + 2yGD + y^2 =$$
$$2AE^2 + 2x^2 + 2CG^2 + 2y^2.$$

***Correspondence to**: *Zhivko Zhelev , Faculty of Economics, Department of Informatics and Mathematics, Trakia University, 6014 Stara Zagora, Bulgaria, 042 699 437, zhelev@uni-sz.bg*

And since the triangles *AOE* and *COG* are right-angle ones, it follows that $AE^2 + y^2 = r^2$ and $CG^2 + x^2 = r^2$, so

$$AF^2 + BF^2 + CF^2 + DF^2 = 2(AE + y^2) + 2(CG^2 + x^2) = 4r^2 \qquad \square$$

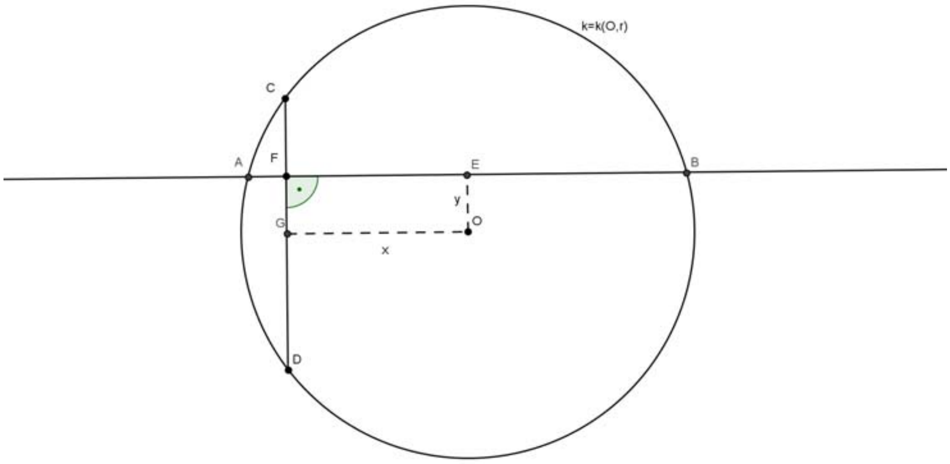

Fig. 1 : The case $F \equiv O$ helps in finding the constant

Prediction is also quite useful in constructing math problems. When constructing and solving problems we can amplify the developing function of prediction and guessing. Obviously, the effectiveness in problem solving depends heavily on prediction and "the prediction itself can be realized by a sequence of elementary steps each one of which is a plausible conclusion." [1, p.15].

Generally, in the literature there is no full and systematic classification of the methods used in mathematics problem solving. One can find different definitions of what "method" means, and, on the other hand, no systematic classification of these methods is available. S. Grozdev [2] for example points out the following methods: induction, proving by admitting the opposite (*reductio ad absurdum*), mathematical induction, substitutions in algebra and geometry, coordinates, vectors, invariants and semi-invariants, etc. He concludes that there is no systematic research on the methods in mathematics problem solving [2].

One of the most successful attempts mathematics to be revealed in the process of its discovery is due to D. Polya (1887-1985). He is first among dozens of mathematicians to write about the *plausible reasoning* when providing a proof in mathematics problem solving. He says that "we provide mathematical knowledge using proofs but we support our conjectures by plausible reasoning" [3].

2. The notion of "heuristic component"

We believe that the basic stage in the act of problem solving is the investigation of what we denote by "heuristic component" of problem solving. What do we mean by that? Here is the following

Definition: *The heuristic component of problem solving is a structure unit in creation of skills in mathematics problem solving. On operational level heuristic component coincides with the heuristic method and on analytical level it appears as a reflection* [4].

To illustrate the importance of prediction and guessing in problem solving, let us consider the following

Problem 2. *On the hypotenuse AB of the isosceles right-angle triangle ABC an arbitrary point M is chosen. Points* G_1 *and* G_2 *are the centroid points to the triangles AMC and BMC respectively. Prove that* $\angle G_1CG_2 > 45^\circ$.

This problem was given to the 8th grade students at one of the rounds of the Bulgarian Mathematics Olympiad in 2010. Using the approach of Kuluitkin [5] and Petrov [6], we shall describe the entire search for solution in three stages.

I stage: Understanding the problem and formulation of the general hypothesis.

Solving the problem starts with understanding the problem. The general hypothesis is built after a thorough analysis of the problem. *The general hypothesis consists of guessing the connections among the groups of subjects involved in the problem*. In the context of the problem above we can say that the inequality $\angle G_1CG_2 > 45^\circ$ can be proved if we pay special attention to the points G_1 and G_2 which are not arbitrary points even though the point *M* is such a point **(Fig. 2)**. Now, since we have an isosceles right-angle triangle given, it can be easily completed to a square. But the square is a symmetric figure and that will allow us to work with angles, distances, etc. much more easily.

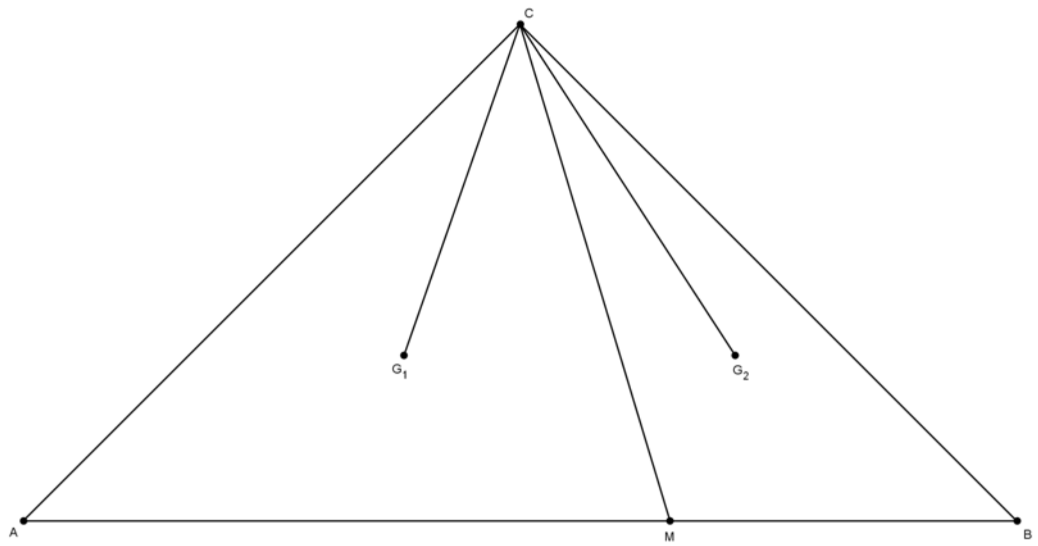

Fig 2 : $\angle G_1MG_2$ should be greater than 45°

II stage: Developing the general hypothesis and constructing a set of ideas for solution

Now, it is more or less clear that in order to solve the problem we should replace the triangle G_1CG_2 with the square $AQCC_1$ **(Fig. 3)**. Then $\angle PQC = \angle G_1CG_2$.

From here on the solution can be done without any obstacles.

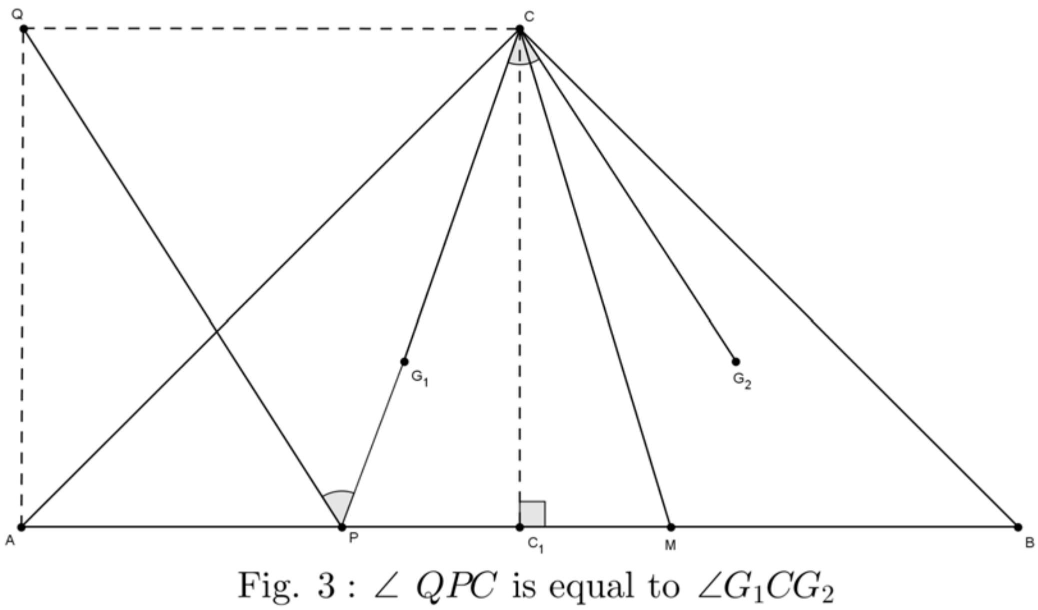

Fig. 3 : $\angle$ QPC is equal to $\angle G_1CG_2$

Of course, the problem can be solved using trigonometry but, unfortunately, trigonometric formulae are not in the curriculum for the 8th grade students.

III stage: Formulation of the basic (specific) hypothesis

To finalize the solution we use the fact that $\angle QPC = \angle G_1CG_2$ and since $P \in AC_1$ it follows that $\angle QPC > 45^\circ \Rightarrow \angle G_1CG_2 > 45^\circ$, q. e. d.

We should mention here that our algorithmic approach in investigating the process of problem solving does not depend on the type of the math problems. To confirm consider the following number theory

Problem 3. *Find all positive integers n such that* $n^5 + 3n + 4$ *is a power of 2.*

The formulation of the general hypothesis here depends on the $n \in \mathbf{Z}^+$. Finding the factors of the polynomial is a fruitful idea here. We have that

$$n^5 + 3n + 4 = 2^k, \; k = 0,1,2,....,$$

On the other side

$$n^5 + 3n + 4 = n^5 + 4n - n + 4 = n(n^4 - 1) + 4(n+1) =$$
$$= n(n-1)(n+1)(n^2+1) + 4(n+1) = (n+1)[n(n-1)(n^2+1) + 4].$$

From here we get that $n = 2^l - 1, \; l = 0,1,2,...$ Precisely this form of *n* is key to the solution. As the second stage of our solution we can point out the divisibility of the polynomial $n(n-1)(n^2+1)$. Obviously it is divided by 4; it is also divided by 3 because $n = 2^l - 1$, and it is also divided by 5. Therefore 60 divides $n(n-1)(n^2+1)$.

On the third stage we formulate our specific hypothesis; i. e. $n(n-1)(n^2+1)$ must be equal to 60. This hypothesis leads us to the only two solutions: $n = 1$ and $n = 3$.

CONCLUSION

In formulation of any mathematical hypothesis, plausible reasoning turns out to be the most important part of the self-regulation at any stage in the act of problem solving. [1]

REFERENCES

1. Petrov, P., Milloushev, V. The Role of Prediction in Mathematics Problem Solving. Prognostic functions of the problem solving methods. Plovdiv University “P. Hilendarski” Scientific Works, 27:13-26, 1990. (Bg)
2. Grozdev, S. For High Achievements in Mathematics. The Bulgarian Experience *(Theory and Practice)*, Sofia, 2007.
3. Polya, D. *Mathematics and Plausible Reasoning: Patterns and Plausible Inference*. Vol. II, Princeton Univ. Press, 1968.
4. Zhelev, Zh., Petrov, P. *The Impact of Reflection on Mathematics Problem Solving*. Trakia Journal of Science, Vol. VIII (3), 2010, p. 1-5.
5. Kuluitkin, Y. *Heuristic methods in the problem solving structure*, Pedagogika, Moscow, 1970. (Rus)
6. Petrov, P. *The problem solving skill in solving school math problems – a fundamental problem in mathematics*, Habilitation Thesis, Stara Zagora, 2011. (Bg)